

NON-EMPTYNESS OF BRILL-NOETHER LOCI OVER VERY GENERAL QUINTIC HYPERSURFACE

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ABSTRACT. In this article we study Brill-Noether loci of moduli space of stable bundles over smooth surfaces. We define Petri map as an analogy with the case of curves. We show the non-emptiness of certain Brill-Noether loci over very general quintic hypersurface in \mathbb{P}^3 , and use the Petri map to produce components of expected dimension.

1. INTRODUCTION

Let X be a smooth, irreducible, projective variety of dimension n over \mathbb{C} , H be an ample divisor on X , and let $\mathcal{M} := \mathcal{M}_{X,H}(r; c_1, \dots, c_s)$ be the moduli space of rank r , H -stable vector bundles E over X with Chern classes $c_i(E) = c_i$, where $s := \min\{r, n\}$. A Brill-Noether locus $B_{r,X,H}^k$ is a closed subscheme of \mathcal{M} whose support consists of points $E \in \mathcal{M}$ such that $h^0(X, E) \geq k + 1$. Göttsche et al ([5]) and M. He ([6]) studied the Brill-Noether loci of stable bundles over \mathbb{P}^2 , and Leyenson ([8], [9]) studied it for $K3$ surfaces. In the case of smooth, projective, irreducible curves C over \mathbb{C} , the Brill-Noether loci of the moduli space, $\text{Pic}^d(C)$, of degree d line bundles on C is well-studied. The questions like non-emptiness, connectedness, irreducibility, singular locus etc of Brill-Noether loci are known when C is a general curve in the sense of moduli (see e.g. [1]). This concept was generalized for vector bundles over curves by Newstead, Teixidor and others. For an account of the results and history in this case see [4] and the references therein.

Recently, in [2], authors have constructed a Brill-Noether loci over higher dimensional varieties under the additional cohomology vanishing assumptions: $H^i(X, E) = 0, \forall i \geq 2$ and for all $E \in \mathcal{M}$. This is a natural generalization of the Brill-Noether loci over the curves for higher dimensional varieties. In [2], [3], authors gave several examples of non-empty Brill-Noether locus, and examples of Brill-Noether locus where “expected dimension” is not same as the exact dimensions. In all these examples, the surfaces and/or varieties chosen have the canonical bundle has no non-zero sections. In this article, we defined “Petri map” over a smooth projective variety with canonical bundle ample, as an analogue of that for curves. Similar to the case of curves, the injectivity of the “Petri map” implies the existence of smooth

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points in Brill-Noether loci. We then use this fact to prove the existence of a smooth point and hence a component of expected dimension in the Brill-Noether loci over a very general quintic hypersurface in \mathbb{P}^3 where the canonical bundle is ample and globally generated.

Notation: We work throughout over the field \mathbb{C} of complex numbers. If X is a smooth, projective variety, we denote by K_X the canonical bundle on X . For a coherent sheaf \mathcal{F} on X , we denote by $H^i(X, \mathcal{F})$ the i -th cohomology group of \mathcal{F} and by $h^i(X, \mathcal{F})$ its (complex) dimension. If V is a vector bundle on X , we denote by V^* the dual of V .

2. BRILL-NOETHER LOCI

In this section we will briefly recall the construction of Brill-Noether loci over higher dimensional varieties, following [2].

Let X be an irreducible, smooth, projective variety of dimension n , and let H be an ample divisor on X . For a torsion free sheaf F over X , let $c_i(F)$ denotes the i -th Chern class of F . Set $\mu(F) = \mu_H(F) := \frac{c_1(F) \cdot H^{n-1}}{\text{rank}(F)}$.

Definition 2.1. A torsion-free sheaf F over X of rank r is called *H-semistable* if for all non-zero subsheaf G of F with $\text{rank}(G) < \text{rank}(F)$, we have

$$\mu(G) \leq \mu(F).$$

We say F is *H-stable* if the above inequality is strict.

Let $\mathcal{M} := \mathcal{M}_{X,H}(r; c_1, \dots, c_s)$ be the moduli space of rank r , H -stable vector bundles E over X with Chern classes $c_i(E) = c_i$, where $s := \min\{r, n\}$. Assume that \mathcal{M} is a fine moduli space, and let $\mathcal{E} \rightarrow \mathcal{M} \times X$ be an universal family such that for any $t \in \mathcal{M}$, $E_t := \mathcal{E}|_{t \times X}$ is a rank r , H -stable bundle over X with Chern classes $c_i(E_t) = c_i$. Choose an effective divisor D on X such that $H^i(X, E_t(D)) = 0, \forall i \geq 1$ and $\forall t \in \mathcal{M}$. Let $\mathcal{D} := \mathcal{M} \times D$ be the product divisor, and $\phi : \mathcal{M} \times X \rightarrow \mathcal{M}$ be the projection map. From the exact sequence

$$0 \rightarrow \mathcal{E} \rightarrow \mathcal{E}(\mathcal{D}) \rightarrow \mathcal{E}(\mathcal{D})/\mathcal{E} \rightarrow 0$$

on $\mathcal{M} \times X$, we get an exact sequence on \mathcal{M} :

$$0 \rightarrow \phi_*(\mathcal{E}) \rightarrow \phi_*(\mathcal{E}(\mathcal{D})) \xrightarrow{\gamma} \phi_*(\mathcal{E}(\mathcal{D})/\mathcal{E}) \rightarrow R^1\phi_*(\mathcal{E}) \rightarrow 0.$$

Note that, γ is a map between two locally free sheaves of ranks $\chi(E_t(D))$ and $\chi(E_t(D)) - \chi(E_t)$ respectively, on \mathcal{M} . For an integer $k \geq -1$, let $B_{r,X,H}^k \subset \mathcal{M}$ be the $(\chi(E_t(D)) - (k+1))$ -th determinantal variety associated to the map γ . Now assume $H^i(X, E_t) = 0, \forall i \geq 2$ and $\forall t \in \mathcal{M}$. Then we have

$$\text{Support}(B_{r,X,H}^k) = \{E \in \mathcal{M} : h^0(X, E) \geq k+1\}.$$

When \mathcal{M} is not a fine moduli space, it is possible to carry out this construction locally and then can be glued together to get a global algebraic object. We summarize the above construction as

Theorem 2.2. ([2], Theorem 2.3) Let X be a smooth, irreducible, projective variety of dimension n , H be a fixed ample divisor on X , and $\mathcal{M} := \mathcal{M}_{X,H}(r; c_1, \dots, c_s)$ be a moduli space of rank r , H -stable vector bundles E on X with fixed Chern classes $c_i(E) = c_i$, $s = \min\{r, n\}$. Assume that for any $E \in \mathcal{M}$, $H^i(X, E) = 0$ for $i \geq 2$. Then for any $k \geq -1$, there exists a determinantal variety $B_{r,X,H}^k \subset \mathcal{M}$ such that

$$\text{Support}(B_{r,X,H}^k) = \{E \in \mathcal{M} : h^0(X, E) \geq k+1\}.$$

Moreover, each non-empty irreducible component of $B_{r,X,H}^k$ has dimension at least

$$\dim(\mathcal{M}) - (k+1)(k+1 - \chi(r; c_1, \dots, c_s))$$

where $\chi(r; c_1, \dots, c_s) := \chi(E_t)$ for any $t \in \mathcal{M}$ and

$$B_{r,X,H}^{k+1} \subset \text{Sing}(B_{r,X,H}^k)$$

whenever $B_{r,X,H}^k \neq \mathcal{M}$.

Definition 2.3. The variety $B_{r,X,H}^k$ is called the k -th Brill-Noether locus of the moduli space \mathcal{M} and the number

$$\rho_{r,X,H}^k := \dim(\mathcal{M}) - (k+1)(k+1 - \chi(r; c_1, \dots, c_s))$$

is called the *generalized Brill-Noether number*.

By the above theorem, the dimension of $B_{r,X,H}^k$ is at least $\rho_{r,X,H}^k$. We call $\rho_{r,X,H}^k$, the *expected dimension* of the Brill-Noether locus $B_{r,X,H}^k$. When there is no confusion about X and H , we will simply denote these by B_r^k and ρ_r^k .

3. PETRI MAP

In this section we will define “Petri Map” for higher dimensional varieties, as an analogue of the one defined for curves. We note that the description of Petri map over curves as given in [4] works for higher dimensional varieties also. For convenience, we recall this description.

Let X be an irreducible, smooth, projective variety of dimension n , H be an ample divisor on X , and let $\mathcal{M} := \mathcal{M}_{X,H}(r; c_1, \dots, c_s)$ be the moduli space of rank r , H -stable vector bundles E over X with Chern classes $c_i(E) = c_i$, where $s := \min\{r, n\}$. The tangent space to \mathcal{M} at a point $E \in \mathcal{M}$ is given by $H^1(X, E \otimes E^*)$, where E^* is the dual of E . A tangent vector to \mathcal{M} at E can be identified with a vector bundle E_ϵ on $X_\epsilon := X \times \text{Spec}(k[\epsilon]/\epsilon^2)$ whose restriction on X is E and it fits into the exact sequence

$$0 \rightarrow E \rightarrow E_\epsilon \rightarrow E \rightarrow 0.$$

We call it a first order deformation of E .

One can give an explicit description of the bundle E_ϵ as follows: Let $\{U_i\}$ be an open cover of X such that $E_i := E|_{U_i}$ is the trivial bundle. Set $U_{ij} := U_i \cap U_j$, and let $\phi_{ij} \in H^0(U_{ij}, E \otimes E^*)$ be the co-boundary map

corresponding to $\phi \in H^1(X, E \otimes E^*)$. Consider the trivial extension of E_i to $U_i \times \text{Spec}(k[\epsilon]/\epsilon^2)$ given by $E_i \oplus \epsilon E_i$. Then the matrix

$$\begin{bmatrix} \text{Id} & 0 \\ \phi_{ij} & \text{Id} \end{bmatrix}$$

will give the gluing data for the bundle E_ϵ .

Assume that a section s of E can be extended to a section of E_ϵ . Then we have local sections $(s'_i) \in H^0(U_i, E_i)$ such that $(s|_{U_i}, s'_i)$ defines a section of E_ϵ . If this is the case, then we have

$$\begin{bmatrix} \text{Id} & 0 \\ \phi_{ij} & \text{Id} \end{bmatrix} \begin{bmatrix} s|_{U_i} \\ s'_i \end{bmatrix} = \begin{bmatrix} s|_{U_j} \\ s'_j \end{bmatrix}.$$

This gives two conditions: $(s|_{U_i})|_{U_{ij}} = (s|_{U_j})|_{U_{ij}}$ and $\phi_{ij}(s) = s'_j - s'_i$. The first condition is automatically satisfied, since s is a global section and from the second condition we see that, in this case, $(\phi_{ij}(s))$ satisfies the co-cycle condition. In other words, $(\phi_{ij}(s))$ is in the kernel of the map

$$H^1(X, E \otimes E^*) \longrightarrow H^1(X, E), \quad (\nu_{ij}) \mapsto (\nu_{ij}(s)).$$

Let $E \in B_r^k, k \geq 0$ and let $s \in H^0(X, E)$. Then the first order deformation of E , as an element of B_r^k , is the subset $\{E_\epsilon : \text{the section } s \text{ can be extended to a section of } E_\epsilon\}$ of $H^1(X, E \otimes E^*)$, i.e.

$$E_\epsilon \in \text{Ker}(H^1(X, E \otimes E^*) \longrightarrow H^1(X, E), \quad (\nu_{ij}) \mapsto (\nu_{ij}(s))).$$

Now assume $E \in B_r^k - B_r^{k+1}$ and let T be the tangent space to B_r^k at the point E . From the discussion above, we have a map

$$\alpha : H^0(X, E) \otimes H^1(X, E \otimes E^*) \longrightarrow H^1(X, E).$$

This induces the map

$$\mu : H^0(X, E) \otimes H^{n-1}(X, K_X \otimes E^*) \longrightarrow H^{n-1}(X, K_X \otimes E \otimes E^*).$$

Note that, T can be identified with $(\text{Im}(\mu))^\perp$. We call the map μ , the *Petri map*.

Remark 3.1. Let X be an irreducible, smooth, projective surface and H be an ample divisor on X . In this case, Petri map, as defined above, is the cup product map

$$(1) \quad \mu : H^0(X, E) \otimes H^1(X, K_X \otimes E^*) \longrightarrow H^1(X, K_X \otimes E \otimes E^*).$$

If E is a smooth point in the moduli space \mathcal{M} , we have

$$\dim(T) = \dim(\mathcal{M}) - h^0(X, E)h^1(X, E) + \dim \text{Ker}(\mu).$$

Thus, if the Petri map is injective, then E is a smooth point of B_r^k and the component of B_r^k through E has the expected dimension.

Remark 3.2. The Petri map can also be derived from [6]. Indeed, by taking $\Lambda = \Lambda' = (H^0(X, E), \text{Id}, E)$ in [6, Corollary 1.6], we see that the above Petri map is dual of the map $\text{Ext}^1(E, E) \rightarrow \text{Hom}(H^0(X, E), H^1(X, E))$ in the given exact sequence.

4. BRILL-NOETHER LOCI OVER QUINTIC HYPERSURFACE

Let $X \subset \mathbb{P}^3$ be a very general smooth, irreducible, projective hypersurface of degree 5. Then $\text{Pic}(X) = \text{Pic}(\mathbb{P}^3) \simeq \mathbb{Z}$, $K_X \simeq \mathcal{O}_X(1)$, $\chi(X, \mathcal{O}_X) = 5$. Let H be a hyperplane class, and $\mathcal{M}(c_2) := \mathcal{M}_{X,H}(2; 3H, c_2)$ be the moduli space of rank 2, H -stable bundles with first Chern class $3H$ and second Chern class c_2 . It is known ([11]) that $\mathcal{M}(c_2)$ is irreducible for $c_2 \geq 14$, generically smooth for $c_2 \geq 21$, and has the expected dimension $4c_2 - 60$. Also note that $H^2(X, E) = 0, \forall E \in \mathcal{M}$. Thus the hypothesis of the Theorem 2.2 is satisfied.

Theorem 4.1. *With the notations as above, $B_2^1 \subset \mathcal{M}(c_2)$ is non-empty for $17 \leq c_2 \leq 32$. Further more, if $27 \leq c_2 \leq 32$, then $B_2^1 - B_2^2$ is non-empty.*

Proof. Let C be an irreducible, smooth, projective curve in the complete linear system $|3H|$. From the known results of Brill-Noether theory for curves (see, e.g. [1]), it is easy to see that there is a globally generated line bundle A on C with $17 \leq \deg(A) \leq 32, h^0(C, A) = 2$.

Fix such a line bundle A on C and consider the elementary transformation

$$0 \rightarrow F \rightarrow H^0(C, A) \otimes \mathcal{O}_X \rightarrow A \rightarrow 0.$$

Then F is a rank two vector bundle on X with $c_1(F) = -3H, c_2(F) = \deg(A)$, $H^0(X, F) = H^1(X, F) = 0$ ([7, Chapter 5, Proposition 5.2.2]). Dualizing the above exact sequence we get

$$(2) \quad 0 \rightarrow H^0(C, A)^* \otimes \mathcal{O}_X \rightarrow F^* \rightarrow \mathcal{O}_C(C) \otimes A^* \rightarrow 0.$$

Thus $h^0(X, F^*) = 2 + h^0(C, \mathcal{O}_C(C) \otimes A^*) \geq 2$, where A^* denotes the dual of A (on C). Set $E := F^*$. We claim that $E \in B_2^1$. For this, it is sufficient to prove that F is H -stable.

If $\mathcal{O}_X(m)$ destabilizes F , then $m \geq -1$. On the other hand, for any line bundle $\mathcal{O}_X(m) \hookrightarrow F$, we have $m \leq 0$ from the construction of F . Also, from $h^0(X, F) = 0$, we see that $m \neq 0$. Thus we are reduced to show that $H^0(X, F \otimes \mathcal{O}_X(1)) = 0$. Now $F \otimes \mathcal{O}_X(1) \simeq F^* \otimes \mathcal{O}_X(-2)$. If $H^0(X, F^* \otimes \mathcal{O}_X(-2)) \neq 0$, we have an inclusion

$$\mathcal{O}_X(2) \hookrightarrow F^*$$

contradicting that $h^0(X, F^*) = 2 + h^0(C, \mathcal{O}_C(C) \otimes A^*)$, as $\mathcal{O}_C(C) \otimes A^*$ is a line bundle of degree at most 28 and hence it can have at most 4 sections.

If $27 \leq c_2 \leq 32$, then we can choose a line bundle A over C such that $h^0(C, A) = 2$ and $h^0(C, \mathcal{O}_C(C) \otimes A^*) = 0$. In this case $h^0(X, F^*) = 2$. This completes the proof. \square

Proposition 4.2. *With the notations as in the beginning of this section, let $E \in B_2^1 - B_2^2 \subset \mathcal{M}(c_2)$, $27 \leq c_2 \leq 32$ be a general element constructed as in the above theorem. Then E is a smooth point in $\mathcal{M}(c_2)$.*

Proof. Since the morphism $\mathcal{M}_{X,H}(2; 3H, c_2) \xrightarrow{\otimes \mathcal{O}_X(-1)} \mathcal{M}_{X,H}(2; H, c_2 - 10)$ is an isomorphism, it is sufficient to prove that $E \otimes \mathcal{O}_X(-1)$ is a smooth

point of $\mathcal{M}_{X,H}(2; H, c_2 - 10)$. Set $F := E \otimes \mathcal{O}_X(-1)$. If F is not a smooth point of $\mathcal{M}_{X,H}(2; H, c_2 - 10)$, we have $H^0(X, \text{Ad}F \otimes K_X) \neq 0$. Thus there is a non-zero map $\phi : F \rightarrow F \otimes K_X$. Two cases can occur:

Case I: ϕ drops rank every where.

Case II: ϕ does not drop rank every where.

Following [10], we call the Case I as singularity of first kind, and the Case II as singularity of second kind. If Case I occurs, then we will have $H^0(X, F) \neq 0$ (see [10]), which is impossible. Also, from [10], we see that the dimension of the singular locus of the second kind is at most 8¹. But the dimension of the globally generated line bundle A on C with $27 \leq \deg(A) \leq 32$, $h^0(C, A) = 2$ and $h^0(C, \mathcal{O}_C(C) \otimes A^*) = 0$ is greater than 8. Hence a general E , as constructed in the previous theorem, will be a smooth point in $\mathcal{M}_{X,H}(2; 3H, c_2)$. \square

Let $E \in B_2^1 - B_2^2$ be a smooth point in $\mathcal{M}(c_2)$, $27 \leq c_2 \leq 32$, as constructed in Theorem 2.2. Existence of such a smooth point is assured by Proposition 4.2. Then E fits into the following exact sequence

$$(3) \quad 0 \rightarrow \mathcal{O}_X^{\oplus 2} \rightarrow E \rightarrow \mathcal{O}_C(C) \otimes A^* \rightarrow 0$$

(see equation (2)). Tensoring this exact sequence by $E^* \otimes K_X$ we get

$$(4) \quad 0 \rightarrow (E^* \otimes K_X)^{\oplus 2} \rightarrow \mathcal{E}nd(E) \otimes K_X \rightarrow \mathcal{O}_C(C) \otimes A^* \otimes (E^* \otimes K_X)|_C \rightarrow 0.$$

Note that $h^0(X, E^* \otimes K_X) = h^2(X, E) = 0$, by our assumption, and since E is a smooth point in the moduli space, $H^0(X, \mathcal{E}nd(E) \otimes K_X) \simeq H^0(X, K_X)$. Thus, by taking cohomology long exact sequence corresponding to the exact sequence (4), we get

$$(5) \quad \begin{aligned} 0 &\rightarrow H^0(X, K_X) \rightarrow H^0(C, \mathcal{O}_C(C) \otimes A^* \otimes (E^* \otimes K_X)|_C) \\ &\rightarrow H^1(X, (E^* \otimes K_X)^{\oplus 2}) \xrightarrow{\eta} H^1(X, \mathcal{E}nd(E) \otimes K_X) \rightarrow \dots \end{aligned}$$

Note that the Petri map μ in (1) is same as the map η above. Now we show that the map η is injective, and this will in turn imply that the bundle E is a smooth point in B_2^1 .

Dualizing the exact sequence (3) and restricting to the curve C , we obtain the exact sequence (on C)

$$0 \rightarrow \mathcal{O}_C(-C) \otimes A \rightarrow E^*|_C \rightarrow \mathcal{O}_C^{\oplus 2} \rightarrow A \rightarrow 0$$

which induces the following short exact sequence (on C)

$$(6) \quad 0 \rightarrow \mathcal{O}_C(-C) \otimes A \rightarrow E^*|_C \rightarrow A^* \rightarrow 0.$$

¹In [10, Corollary 5.1], the authors have estimated the bound for singularity of second kind as ≤ 13 . But in [11, Lemma 10.1], the authors have improved the above bound and shown that it is ≤ 8

Tensoring the sequence (6) by $\mathcal{O}_C(C) \otimes A^* \otimes K_X|_C$ and taking cohomology long exact sequence we get

$$\begin{aligned} 0 \rightarrow H^0(C, K_X|_C) &\rightarrow H^0(C, \mathcal{O}_C(C) \otimes A^* \otimes (E^* \otimes K_X)|_C) \\ &\rightarrow H^0(C, \mathcal{O}_C(C) \otimes A^{*\otimes 2} \otimes K_X|_C) \rightarrow \dots \end{aligned}$$

By our assumption on A , $H^0(C, \mathcal{O}_C(C) \otimes A^*) = 0$. Since $\deg(A^* \otimes K_X|_C) < 0$, we have $H^0(C, \mathcal{O}_C(C) \otimes A^{*\otimes 2} \otimes K_X|_C) = 0$. Thus from the above exact sequence $H^0(C, K_X|_C) \simeq H^0(C, \mathcal{O}_C(C) \otimes A^* \otimes (E^* \otimes K_X)|_C)$. It is easy to see that $H^0(X, K_X) \simeq H^0(C, K_X|_C)$. Consequently, the map η in (5) is injective.

We summarize the above discussion as

Theorem 4.3. $B_2^1 \subset \mathcal{M}(c_2)$, $27 \leq c_2 \leq 32$, contains a smooth point, and hence the irreducible component containing it has the expected dimension.

5. NON-EMPTYNESS OF B_2^0

Let X and $\mathcal{M}(c_2)$ be as in the previous section. Let $\text{Hilb}^{c_2}(X)^{\text{lci}}$ denotes the open subscheme of the Hilbert scheme $\text{Hilb}^{c_2}(X)$ consisting of length c_2 subschemes of X which are locally complete intersections. Given any point $Z \in \text{Hilb}^{c_2}(X)^{\text{lci}}$, we have an exact sequence

$$0 \rightarrow \mathcal{O}_X \rightarrow E \rightarrow \mathcal{I}_Z \otimes \mathcal{O}_X(3) \rightarrow 0$$

where \mathcal{I}_Z is the ideal sheaf of Z and E is a rank two torsion-free sheaf on X . The space of such (isomorphic classes of) extensions is parametrized by $\mathbb{P}\text{Ext}^1(\mathcal{I}_Z \otimes \mathcal{O}_X(3), \mathcal{O}_X)$. By duality, $\text{Ext}^1(\mathcal{I}_Z \otimes \mathcal{O}_X(3), \mathcal{O}_X) = H^1(X, \mathcal{I}_Z \otimes \mathcal{O}_X(4))^*$. Since $h^0(X, \mathcal{O}_X(4)) = 35$, for a general element of length $c_2 \geq 36$ in $\text{Hilb}^{c_2}(X)^{\text{lci}}$, we have $h^0(X, \mathcal{I}_Z \otimes \mathcal{O}_X(4)) = 0$. Thus a general element in $\text{Hilb}^{c_2}(X)^{\text{lci}}$, $c_2 \geq 36$, satisfies the Cayley-Bacharach property for $\mathcal{O}_X(4)$, and consequently, we get that the corresponding extension E is locally free. Also any such vector bundle E is H -stable. Thus for $c_2 \geq 36$, $B_2^0 \subset \mathcal{M}(c_2)$ is non-empty.

The following two results give a bound for $\dim B_2^0$.

Lemma 5.1. ([12, Proposition 1.1]²) *With the notations as above, for $c_2 \geq 36$, $\dim B_2^0 \leq 3c_2 - 36$.*

Lemma 5.2. ([10, Corollary 3.1]) *With the notations as above, every irreducible component of B_2^0 has dimension $\geq 3c_2 - h^0(X, \mathcal{O}_X(3) \otimes K_X) - 1$.*

Since $h^0(X, \mathcal{O}_X(3) \otimes K_X) = 35$, we see that, for $c_2 \geq 36$, every irreducible component of B_2^0 has dimension exactly $3c_2 - 36$. On the other hand, the Brill-Noether number $\rho_2^0 = 3c_2 - 36$. We summarize the above discussion as

²In [12], Proposition 1.1 was proved with the assumption that $c_1(E) = H$, and the bound author got there is $\dim B_2^0 \leq 3c_2 - 11$, for $c_2 \geq 10$. The same calculation, in this case, gives us the stated bound

Proposition 5.3. *With the notations as above, $B_2^0 \subset \mathcal{M}(c_2)$ is non-empty for $c_2 \geq 36$, and every irreducible component of B_2^0 has the expected dimension.*

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REFERENCES

- [1] Arbarello, E.; Cornalba, M.; Griffiths, P. A.; Harris, J.: *Geometry of algebraic curves. Vol. I.* Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 267. Springer-Verlag, New York, 1985.
- [2] Costa, Laura; Miró-Roig, Rosa Maria: *Brill-Noether theory for moduli spaces of sheaves on algebraic varieties.* Forum Math. 22 (2010), no. 3, 411-432.
- [3] Costa, L.; Miró-Roig, R. M: *Brill-Noether theory on Hirzebruch surfaces.* J. Pure Appl. Algebra 214 (2010), no. 9, 1612-1622.
- [4] I. Grzegorzcyk; M. Teixidor i Bigas: *Brill-Noether theory for stable vector bundles.* arXiv:0801.4740
- [5] Göttsche, L; Hirschowitz, A: *Weak Brill-Noether for vector bundles on the projective plane.* Algebraic geometry (Catania, 1993/Barcelona, 1994), 63-74, Lecture Notes in Pure and Appl. Math., 200, Dekker, New York, 1998.
- [6] He, Min: *Espaces de modules de systèmes cohérents.* Internat. J. Math. 9 (1998), no. 5, 545-598.
- [7] Huybrechts, Daniel; Lehn, Manfred: *The geometry of moduli spaces of sheaves.* Second edition. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 2010.
- [8] Leyenson, M: *On the Brill-Noether theory for K3 surfaces.* Cent. Eur. J. Math. 10 (2012), no. 4, 1486-1540.
- [9] Leyenson M.: *On the Brill-Noether theory for K3 surfaces, II.* <https://arxiv.org/abs/math/0602358>
- [10] Mestrano, Nicole; Simpson, Carlos: *Obstructed bundles of rank two on a quintic surface.* Internat. J. Math. 22 (2011), no. 6, 789-836.
- [11] N. Mestrano, C. Simpson: *Irreducibility of the moduli space of stable vector bundles of rank two and odd degree on a very general quintic surface.* arXiv. <https://arxiv.org/abs/1302.3736>
- [12] P. Nijse: *The irreducibility of the moduli space of stable vector bundles of rank 2 on a quintic in \mathbb{P}^3 .* arXiv. <https://arxiv.org/abs/alg-geom/9503012>

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